THE CRITICAL GROUP OF A DIRECTED GRAPH

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Abstract. For a finite, directed graph $G = (V, E)$ we define the critical group $\mathcal{K}(G)$ to be the cokernel of the transpose of the Laplacian matrix of G acting on $\mathbf{Z}^V$, and $K(G)$ to be its torsion subgroup. This generalizes the case of undirected graphs studied by Bacher, de la Harpe and Nagnibeda, and by Biggs. We prove a variety of results about these critical groups, among which are: that $\mathcal{K}(G/\pi)$ is a subgroup of $\mathcal{K}(G)$ when π is an equitable partition and G is strongly connected; that $K(G)$ depends only on the graphic matroid of G when G is undirected; that there is no 'natural' bijection between spanning trees of G and $K(G)$ when G is undirected, even though these sets are equicardinal; and that the 'dollar game' of Biggs can be generalized slightly to provide a combinatorial interpretation for the elements of $K(G)$ when G is strongly connected.

1. Introduction.

We use the word *graph* to refer to a finite, possibly directed, multigraph. If a graph is undirected then we consider each of its edges to represent a pair of directed edges with opposite orientations. (In particular, an undirected loop represents two directed loops.) The *adjacency matrix* $A(G)$ of a graph $G = (V, E)$ is indexed by $V \times V$, with vw–entry A_{vw} being the number of directed edges of G with initial vertex v and terminal vertex w. The matrix $\Delta(G)$ is indexed by $V \times V$, with diagonal entry $\Delta_{vv} := \sum_{w \in V} A_{vw}$ being the outdegree of the vertex v, and with zero off-diagonal entries. The *Laplacian matrix* of G is $Q(G) := \Delta(G) - A(G)$.

The (*full*) *critical group* $\mathcal{K}(G)$ of $G = (V, E)$ is the cokernel of the transpose of its Laplacian matrix acting on $\mathbf{Z}^V$; that is,

$$\mathcal{K}(G) := \mathbf{Z}^V / Q^\dagger(G)\mathbf{Z}^V.$$

This is a finitely generated abelian group. Also, we define the *reduced critical group* $K(G)$ of G to be the torsion subgroup of $\mathcal{K}(G)$; that is,

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the subgroup of $\mathcal{K}(G)$ consisting of all elements of finite order. For a connected, undirected graph G, this $K(G)$ is the 'Jacobian group' defined by Bacher, de la Harpe, and Nagnibeda [1] and studied further by Biggs [3, 4, 5] under the term 'critical group'. Our goal here is to investigate the relationship between the combinatorial structure of graphs and the algebraic structure of their critical groups as generally as possible. Since the Laplacian matrix is insensitive to loops in a graph, we might as well restrict attention to loopless graphs; it is only in Section 9, however, that we really require this restriction.

Section 2 contains some preliminary observations. In Section 3, we determine the rank of $\mathcal{K}(G)$ combinatorially. In Section 4, we consider the minimal number of generators of $K(G)$; this is a much more difficult invariant of G than the rank of $\mathcal{K}(G)$, and we obtain only a weak upper bound for it. In Sections 5 and 6, we show that $\mathcal{K}(G/\pi)$ is (isomorphic to) a subgroup of $\mathcal{K}(G)$ when π is an equitable partition of a strongly connected graph G. In Section 7, we prove some isomorphism theorems, with the consequence that for an undirected graph G, the reduced critical group $K(G)$ depends only on the graphic matroid of G. In Section 8, we show that, for a connected undirected graph G, there is no 'natural' bijection between $K(G)$ and the set of spanning trees of G, even though these sets are equicardinal. In Section 9, we revisit the 'dollar game' of Biggs [3, 4, 5] in the more general setting of strongly connected graphs. The theory is almost the same as for undirected graphs, with one interesting new complication. Throughout the paper, we indicate various conjectures and open problems.

2. Preliminaries.

Let G be a graph with $n(G)$ vertices. By the structure theorem for finitely generated abelian groups, there are nonnegative integers g_1, ..., g_n such that g_i divides g_{i+1} for each $1 \leq i \leq n-1$ and

$$\mathcal{K}(G) \simeq (\mathbf{Z}/g_1\mathbf{Z}) \oplus (\mathbf{Z}/g_2\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/g_n\mathbf{Z}).$$

(Of course, $\mathbf{Z}/1\mathbf{Z} = 0$ is the trivial group and $\mathbf{Z}/0\mathbf{Z} = \mathbf{Z}$.) These integers are computed by reducing $Q(G)$ to its Smith normal form. Say that two n-by-n integer matrices M and M' are *equivalent*, denoted by $M \approx M'$, if and only if there exist n-by-n integer matrices L and N with determinant ± 1 such that $LMN = M'$. That is, M can be transformed to M' by applying elementary row and column operations which are invertible over the integers. It is not difficult to verify that every n-by-n integer matrix M is equivalent to a diagonal matrix $\mathrm{diag}(g_1, g_2, \ldots, g_n)$ of nonnegative integers such that g_i divides g_{i+1} for each $1 \leq i \leq n-1$,

and that this matrix $\mathrm{snf}(M)$ is determined uniquely by M. This is the *Smith normal form* of M.

We define the *dual critical group* of G to be

$$\mathcal{K}^*(G) := \mathbf{Z}^V / Q(G)\mathbf{Z}^V,$$

the cokernel of the Laplacian of G acting on $\mathbf{Z}^V$. It is clear that for any square integer matrix M, $\mathrm{snf}(M^\dagger) = \mathrm{snf}(M)$, from which Proposition 2.1 follows.

Proposition 2.1. *For any graph G, $\mathcal{K}^*(G)$ is isomorphic to $\mathcal{K}(G)$.*

Of course, G is undirected if and only if $Q^\dagger = Q$, in which case $\mathcal{K}^*(G) = \mathcal{K}(G)$. In general, however, the isomorphism in Proposition 2.1 depends on a choice of matrices L and N such that $LQN = \mathrm{snf}(Q) = N^\dagger Q^\dagger L^\dagger$, and hence is not natural. These dual critical groups will be useful in Section 6.

If G is a graph with weak components $G_1, \ldots, G_c$, then $Q(G) = Q(G_1) \oplus \cdots \oplus Q(G_c)$ is block-diagonal, from which Proposition 2.2 follows.

Proposition 2.2. *If G is a graph with weak components $G_1, \ldots, G_c$, then*

$$\mathcal{K}(G) = \mathcal{K}(G_1) \oplus \cdots \oplus \mathcal{K}(G_c).$$

Proposition 2.3 is essentially the Matrix-Tree Theorem; see Theorem 6.3 of Biggs [2] or Theorem 7.3 of Biggs [4].

Proposition 2.3. *If G is connected and undirected then the order of $K(G)$ is $\kappa(G)$, the number of spanning trees of G.*

We use the slightly odd but convenient notations M_{vw} for the (v, w)-entry of a V-by-V matrix M, but $x(v)$ for the v-th entry of a V-indexed vector $\mathbf{x}$. Also, $\mathbf{1}$ denotes the all-ones vector, and $\mathbf{0}$ denotes the zero vector.

3. The torsion-free part of $\mathcal{K}(G)$.

For each natural number g, let $\mu_g(G)$ denote the multiplicity with which g occurs on the diagonal of $\mathrm{snf}(Q^\dagger(G))$. Thus, $\mu_0(G)$ is the rank of $\mathcal{K}(G)$, so that $\mathcal{K}(G) \simeq K(G) \oplus \mathbf{Z}^{\mu_0(G)}$. In this section we determine the combinatorial meaning of $\mu_0(G)$ for any graph G.

Since

$$(\mathbf{Z}/g\mathbf{Z}) \otimes \mathbf{R} = \begin{cases} \mathbf{R} & \text{if } g = 0, \\ 0 & \text{if } g \neq 0, \end{cases}$$

for all nonnegative integers g, and since tensor product distributes across direct sums, we see that

$$\mu_0(G) = \dim_{\mathbf{R}} \mathcal{K}(G) \otimes \mathbf{R} = \dim_{\mathbf{R}} \mathbf{R}^V/Q^\dagger \mathbf{R}^V = \dim_{\mathbf{R}} \ker(Q^\dagger).$$

The results of this section, determining $\dim_{\mathbf{R}} \ker(Q^\dagger)$ combinatorially, are well-known, but we repeat the short proofs for completeness and the readers' convenience.

Lemma 3.1. *Let G be a strongly connected graph.*
(a) *The kernel of Q acting on $\mathbf{R}^V$ is $\mathbf{R}\mathbf{1}$, the span of the all-ones vector.*
(b) *If T is a diagonal matrix with nonnegative real entries, then either $T = O$ or $Q + T$ is invertible over $\mathbf{R}$.*

Proof. We prove (a) and (b) together by showing that for T as in part (b) and $\mathbf{z} \neq \mathbf{0}$, if $(Q+T)\mathbf{z} = \mathbf{0}$ then $T = O$ and $\mathbf{z} = c\mathbf{1}$ for some $c \in \mathbf{R}$. With these hypotheses, choose any vertex $v \in V$ such that $|z(v)| > 0$ is maximum. Then, since $(\Delta + T)\mathbf{z} = A\mathbf{z}$ we see that

$$(\Delta_{vv} + T_{vv})z(v) = \sum_{w \in V} A_{vw} z(w).$$

Since there are Δ_{vv} terms on the right side (considering A_{vw} as the multiplicity of the term $z(w)$) and each of these has absolute value at most $|z(v)|$, it follows that $T_{vv} = 0$ and $z(w) = z(v)$ for all $w \in V$ such that $A_{vw} \neq 0$. Now, we may repeat this argument with any such vertex w in place of v, *et cetera*. Since G is strongly connected, it follows that $T = O$ and $\mathbf{z} = z(v)\mathbf{1}$, completing the proof. ☐

Proposition 3.2. *Let G be a strongly connected graph. There is a unique vector $\mathbf{h} \in \mathbf{R}^V$ such that $Q^\dagger \mathbf{h} = \mathbf{0}$, the entries of $\mathbf{h}$ are positive integers, and $\gcd\{h(v) : v \in V\} = 1$. Moreover, $\ker(Q^\dagger) = \mathbf{R}\mathbf{h}$.*

Proof. If G has a single vertex then the result is trivial, so assume $n(G) \geq 2$. Now, since G is strongly connected, the matrix Δ is invertible. Since $\ker(Q)$ is one-dimensional, $\ker(Q^\dagger)$ is also one-dimensional; hence there is a unique (nonzero) vector $\mathbf{z}$ such that $Q^\dagger \mathbf{z} = \mathbf{0}$ and $\mathbf{1}^\dagger \Delta \mathbf{z} = 1$. Then $(\Delta\mathbf{z})^\dagger \Delta^{-1} A = \mathbf{z}^\dagger A = (\Delta\mathbf{z})^\dagger$, so $\Delta\mathbf{z}$ is the stationary distribution of the Markov chain represented by the stochastic matrix $\Delta^{-1}A$. Since G is strongly connected, every state of this Markov chain is recurrent, so every entry of $\mathbf{z}$ is positive. Since $\mathbf{z}$ solves the system $Q^\dagger \mathbf{z} = \mathbf{0}$, which has integer coefficients, every entry of $\mathbf{z}$ is rational. Finally, there is a unique positive integer multiple of $\mathbf{z}$ which gives the vector $\mathbf{h}$ with the desired properties. ☐

For a strongly connected graph G, the vector $\mathbf{h}$ defined in Proposition 3.2 will be significant for several results in what follows. We refer to

$h(v)$ as the *activity* of the vertex $v \in V$, for reasons which will be seen in Section 9.

For S a strong component of G, if $\mathbf{x} \in \mathbf{R}^V$ then let $\mathbf{x}|_S$ be the restriction of $\mathbf{x}$ to $V(S)$, and if M is a V-by-V matrix then let $M|_S$ denote the submatrix of M indexed by rows and columns from $V(S)$.

Lemma 3.3. *For any graph G, if $\mathbf{z} \in \ker(Q^\dagger)$ and S is a non-terminal strong component of G, then $\mathbf{z}|_S = \mathbf{0}$.*

Proof. Let $S_1, \ldots, S_c$ be a list of the strong components of G such that if there is a directed edge from $v \in S_i$ to $w \in S_j$, then $i \leq j$. We prove the claim by induction on $1 \leq j \leq c$. For the basis of induction ($j = 1$), and for the induction step ($j \geq 2$), we may assume that $\mathbf{z}|_{S_i} = \mathbf{0}$ for all $1 \leq i < j$ such that S_i is a non-terminal strong component of G. If S_j is a terminal strong component of G then there is nothing to prove. Otherwise, there is at least one directed edge with initial vertex in S_j and terminal vertex not in S_j. Therefore, $Q(G)|_{S_j} = Q(S_j) + T$ for some nonzero diagonal matrix T of nonnegative integers. Now, since $\mathbf{z}|_{S_i} = \mathbf{0}$ for all $1 \leq i < j$ such that S_i non-terminal,

$$\mathbf{0} = (Q^\dagger(G)\mathbf{z})|_{S_j} = (Q^\dagger(S_j) + T)(\mathbf{z}|_{S_j}),$$

and by Lemma 3.1(b) we conclude that $\mathbf{z}|_{S_j} = \mathbf{0}$. □

Theorem 3.4. *Let G be any graph, and let the terminal strong components of G be $S_1, \ldots, S_t$. Then*

$$\ker(Q^\dagger(G)) \simeq \ker(Q^\dagger(S_1)) \oplus \cdots \oplus \ker(Q^\dagger(S_t)).$$

Proof. Let $\mathbf{z} \in \ker(Q^\dagger(G))$. By Lemma 3.3, $\mathbf{z}|_S = \mathbf{0}$ if S is not a terminal strong component of G. Hence, if S is a terminal strong component of G, then

$$\mathbf{0} = (Q^\dagger \mathbf{z})|_S = Q^\dagger(S)(\mathbf{z}|_S),$$

so that $\mathbf{z}|_S \in \ker(Q^\dagger(S))$. Conversely, if $\mathbf{z}_i \in \ker(Q^\dagger(S_i))$ for each terminal strong component of G then the $\mathbf{z} \in \mathbf{R}^V$ defined by

$$\mathbf{z}|_S := \begin{cases} \mathbf{z}_i & \text{if } S = S_i \text{ for some } 1 \leq i \leq t, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

is in the kernel of $Q^\dagger(G)$. □

Corollary 3.5. *For any graph G, $\mu_0(G)$ is the number of terminal strong components of G.*

4. The minimal number of generators of $K(G)$.

Let $\nu(G)$ denote the minimal number of generators of the reduced critical group of G, so $K(G)$ is the direct sum of $\nu(G)$ nontrivial finite cyclic groups. This is a rather difficult numerical invariant of G, as it depends on the arithmetic properties of $Q(G)$. Clearly, $\mu_1(G) + \nu(G) + \mu_0(G) = n(G)$. We give an upper bound on $\nu(G)$ by proving a combinatorial lower bound on $\mu_1(G)$. It must be admitted, however, that this bound is generally quite weak.

Let G be a weakly connected graph, with Laplacian matrix Q. A *reduction sequence* in G is a sequence $(v_1, w_1), \ldots, (v_s, w_s)$ of pairs of vertices such that:
- the vertices $v_1, \ldots, v_s$ are pairwise distinct,
- the vertices $w_1, \ldots, w_s$ are pairwise distinct,
- for each $1 \leq i \leq s$, $Q_{v_i w_i} \in \{-1, 1\}$, and
- for each $2 \leq j \leq s$, either $Q_{v_i w_j} = 0$ for all $1 \leq i < j$, or $Q_{v_j w_i} = 0$ for all $1 \leq i < j$.

Let $\sigma(G)$ denote the maximum length of a reduction sequence in G.

Proposition 4.1. *If G is a weakly connected graph, then $\mu_1(G) \geq \sigma(G)$, and hence $\nu(G) \leq n(G) - \mu_0(G) - \sigma(G)$.*

Proof. Let $(v_1, w_1), \ldots, (v_s, w_s)$ be a reduction sequence in G, and let Q be the Laplacian matrix of G.

The first claim is that we may apply elementary row and column operations to Q, involving only rows $v_1, \ldots, v_s$ and columns $w_1, \ldots, w_s$, so that these rows and columns of the resulting matrix induce an s-by-s identity matrix. We prove this by induction on s, the basis $s = 1$ being clear. For the induction step, since $(v_1, w_1), \ldots, (v_{s-1}, w_{s-1})$ is a reduction sequence of length $s - 1$, the induction hypothesis gives elementary operations involving only rows $v_1, \ldots, v_{s-1}$ and columns $w_1, \ldots, w_{s-1}$, which, when applied to Q, result in a matrix Q' in which these rows and columns induce an $(s-1)$-square identity matrix. By the last condition defining a reduction sequence, we have either $Q_{v_i w_s} = 0$ for all $1 \leq i < s$, or $Q_{v_s w_i} = 0$ for all $1 \leq i < s$. Examining the way in which Q' was obtained from Q, we see that either $Q'_{v_i w_s} = 0$ for all $1 \leq i < s$, or $Q'_{v_s w_i} = 0$ for all $1 \leq i < s$. Now, elementary row or column operations can be used to cancel any nonzero entries $Q'_{v_i w_s} \neq 0$ or $Q'_{v_s w_i} \neq 0$ with $1 \leq i < s$, and the value of $Q'_{v_s w_s}$ is left unchanged. Finally, multiplying row v_s by -1 if necessary produces the s-by-s identity submatrix, as claimed.

Having produced this s-by-s identity submatrix, we use elementary column operations to zero out all entries in rows $v_1, \ldots, v_s$ except for the 1s in the (v_i, w_i) positions $(1 \leq i \leq s)$. Then we use elementary row

operations to zero out all entries in columns $w_1, \ldots, w_s$ except for the 1s in the (v_i, w_i) positions ($1 \leq i \leq s$). The result is a matrix, equivalent to Q, which is also equivalent to a matrix with the block structure $I_s \oplus M$ for some $(n-s)$-square matrix M. Therefore, 1 occurs at least s times in the Smith normal form of Q. Considering a reduction sequence of maximum length $s = \sigma(G)$ completes the proof. □

The bound of Proposition 4.1 is likely to be very far from the true value of $\nu(G)$, since it makes no use of the arithmetic structure of $Q(G)$. With this in mind, here are a few conjectures. Let $\mathcal{G}(n,p)$ denote a random simple, undirected graph with n vertices and edge-probability $0 \leq p \leq 1$. As is is well-known (see Theorem 4.3.1 of Palmer [8]), if $p(n) > c\log(n)/n$ with $c > 1$ then, as $n \to \infty$, the probability that $\mathcal{G}(n,p)$ is connected converges to 1.

Conjecture 4.2. *If $c > 1$ and $c\log(n)/n < p(n) < 1 - o(\log(n)/n)$ then, as $n \to \infty$, the probability that $K(\mathcal{G}(n,p))$ is cyclic converges to 1.*

That is, the conjecture is that almost every connected undirected simple graph has a cyclic reduced critical group. (The edge-probability must be bounded away from 1 to avoid complete graphs and complete multipartite graphs, but I don't really know what the 'right' bound should be.) There is some experimental evidence for this, but it is not extensive. The following weak form is probably more accessible.

Conjecture 4.3. *If $c > 1$ and $c\log(n)/n < p(n) < 1 - o(\log(n)/n)$ then, as $n \to \infty$, the expected value of $\nu(\mathcal{G}(n,p))$ remains bounded.*

By considering Smith normal forms and using Proposition 2.3, it is easy to see that for a connected undirected graph G, if $\kappa(G)$ is square-free then $K(G)$ is cyclic (or trivial). Since the density of square-free natural numbers is asymptotically $6/\pi^2$, this motivates the third conjecture.

Conjecture 4.4. *If $c > 1$ and $c\log(n)/n < p(n) < 1 - o(\log(n)/n)$ then, as $n \to \infty$, the probability that $\kappa(\mathcal{G}(n,p))$ is square-free is $(1 - o(1))6/\pi^2$.*

5. Equitable partitions of graphs.

See Chapter 5 of Godsil [6] for further development and application of the theory of equitable partitions of undirected graphs.

Consider a graph G, and let $\pi = (\pi_1, \ldots, \pi_p)$ be an ordered partition of V into pairwise disjoint nonempty blocks. The partition π is *equitable* for G provided that there exist nonnegative integers F_{ij} and R_{ij} for all

$1 \leq i, j \leq p$ such that every vertex in π_i is the initial vertex of exactly F_{ij} directed edges of G which have their terminal vertices in π_j, and every vertex in π_j is the terminal vertex of exactly R_{ij} directed edges of G which have their initial vertices in π_i. (The letters F and R are mnemonic for 'forward' and 'reverse', respectively.) These integers define $p \times p$ matrices F and R, and we regard F as the adjacency matrix of a graph G/π on the vertex-set $\{1, \ldots, p\}$, called the *quotient of G by π*. It will be convenient to use the notations A, Δ and Q for $A(G)$, $\Delta(G)$, and $Q(G)$, and to use F, D and $\widehat{Q}$ for $A(G/\pi)$, $\Delta(G/\pi)$, and $Q(G/\pi)$.

For π an equitable partition of G, let P be the matrix indexed by $V \times \{1, \ldots, p\}$, with entries

$$P_{vi} := \begin{cases} 1 & \text{if } v \in \pi_i, \\ 0 & \text{if } v \notin \pi_i. \end{cases}$$

Then $B := P^\dagger P$ is the invertible $p \times p$ diagonal matrix with entries $B_{ii} = \#\pi_i$ for each $1 \leq i \leq p$. For each $1 \leq i, j \leq p$, by counting in two ways the directed edges of G with initial vertex in π_i and terminal vertex in π_j we see that $B_{ii} F_{ij} = R_{ij} B_{jj}$, yielding the matrix equations $BF = RB$ and $B^{-1}R = FB^{-1}$. For any vertex $v \in \pi_i$ we have

$$\Delta_{vv} = \sum_{w \in V} A_{vw} = \sum_{j=1}^{p} F_{ij} = D_{ii},$$

or, in matrix form, $\Delta = PDB^{-1}P^\dagger$. Therefore, $\Delta P = PD$. Also, for $v \in \pi_i$, and any $1 \leq j \leq p$,

$$(AP)_{vj} = \sum_{w \in \pi_j} A_{vw} = F_{ij} = (PF)_{vj},$$

so that $AP = PF$. It follows that $QP = P\widehat{Q}$. Finally, consider any $1 \leq i \leq p$ and $v \in \pi_j$. Then

$$(B^{-1}P^\dagger A)_{iv} = B_{ii}^{-1} \sum_{w \in \pi_i} A_{wv} = B_{ii}^{-1} R_{ij} = F_{ij} B_{jj}^{-1} = (FB^{-1}P^\dagger)_{iv},$$

so that $B^{-1}P^\dagger A = FB^{-1}P^\dagger$. Also, since

$$B^{-1}P^\dagger \Delta = B^{-1}P^\dagger PDB^{-1}P^\dagger = DB^{-1}P^\dagger,$$

we conclude that $B^{-1}P^\dagger Q = \widehat{Q}B^{-1}P^\dagger$.

6. Critical groups of graph quotients.

We continue with the notation of the previous section. The matrix P defines a group homomorphism $P : \mathbf{Z}^p \to \mathbf{Z}^V$, and $P\widehat{Q}\mathbf{Z}^p = QP\mathbf{Z}^V \subseteq Q\mathbf{Z}^V$. Therefore, P induces a homomorphism

$$\rho : \mathcal{K}^*(G/\pi) \longrightarrow \mathcal{K}^*(G)$$

between the dual critical groups, which is well-defined by $\rho(\mathbf{x}+\widehat{Q}\mathbf{Z}^p) := P\mathbf{x} + Q\mathbf{Z}^V$.

Theorem 6.1 was inspired by Theorem 10.2 of Biggs [4].

Theorem 6.1. *Let G be a strongly connected graph, and let π be an equitable partition of G. Then the natural homomorphism $\rho : \mathcal{K}^*(G/\pi) \to \mathcal{K}^*(G)$ is injective.*

Proof. To show that $\rho : \mathcal{K}^*(G/\pi) \to \mathcal{K}^*(G)$ is injective, we must show that if $\mathbf{x} \in \mathbf{Z}^p$ is such that $P\mathbf{x} \in Q\mathbf{Z}^V$, then $\mathbf{x} \in \widehat{Q}\mathbf{Z}^p$. Accordingly, assume that $\mathbf{x} \in \mathbf{Z}^p$ and $\mathbf{v} \in \mathbf{Z}^V$ satisfy $P\mathbf{x} = Q\mathbf{v}$. Let $\mathbf{y} := B^{-1}P^\dagger\mathbf{v}$. Then

$$\mathbf{x} = B^{-1}P^\dagger P\mathbf{x} = B^{-1}P^\dagger Q\mathbf{v} = \widehat{Q}B^{-1}P^\dagger\mathbf{v} = \widehat{Q}\mathbf{y}.$$

The entries of $\mathbf{y}$ are rational numbers, but we need to find $\mathbf{u} \in \mathbf{Z}^p$ such that $\mathbf{x} = \widehat{Q}\mathbf{u}$. To do this, notice that

$$Q\mathbf{v} = P\mathbf{x} = P\widehat{Q}\mathbf{y} = QP\mathbf{y},$$

so that $Q(\mathbf{v} - P\mathbf{y}) = \mathbf{0}$. Since G is strongly connected, the kernel of Q is $\mathbf{R1}$, and this implies that $\mathbf{v} - P\mathbf{y} = c\mathbf{1}$ for some $c \in \mathbf{R}$. Now $\mathbf{v} = P\mathbf{y} + c\mathbf{1} = P(\mathbf{y} + c\mathbf{1})$, and since every entry of $\mathbf{v}$ is an integer, every entry of $\mathbf{u} := \mathbf{y} + c\mathbf{1}$ is an integer. Finally, since $\widehat{Q}\mathbf{1} = \mathbf{0}$ it follows that $\mathbf{x} = \widehat{Q}\mathbf{y} = \widehat{Q}\mathbf{u}$, which shows that $\mathbf{x} \in \widehat{Q}\mathbf{Z}^p$ and completes the proof. □

Of course, under the hypotheses of Theorem 6.1, the natural homomorphism

$$\rho : K^*(G/\pi) \longrightarrow K^*(G)$$

is also injective.

Example 6.2. The hypothesis that G is strongly connected can not be dropped from Theorem 6.1, as the following example shows. Let

$$Q = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \widehat{Q} = \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}.$$

Then Q is the Laplacian matrix of a weakly connected graph G with vertex-set $\{1, 2, 3\}$ and $\widehat{Q}$ is the Laplacian of the quotient of G by the

equitable partition $\pi = \{\{1\},\{2,3\}\}$ of G. However, by computing the Smith normal forms of Q and $\widehat{Q}$ one sees that

$$\mathcal{K}^*(G) \simeq \mathbf{Z} \oplus \mathbf{Z} \qquad \text{and} \qquad \mathcal{K}^*(G/\pi) \simeq (\mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}.$$

Hence, the homomorphism $\rho : \mathcal{K}^*(G/\pi) \to \mathcal{K}^*(G)$ is not injective in this case.

Example 6.3. With the hypotheses of Theorem 6.1, $K^*(G/\pi)$ is regarded as a subgroup of $K^*(G)$ *via* the natural inclusion ρ. However, $K^*(G/\pi)$ might not be a direct summand of $K^*(G)$, as the following example shows. The nine-cycle C_9 has an equitable partition π for which the quotient graph C_9/π is the three-cycle C_3, but $K^*(C_3) \simeq \mathbf{Z}/3\mathbf{Z}$ is not a direct summand of $K^*(C_9) \simeq \mathbf{Z}/9\mathbf{Z}$.

7. Matroid invariance of $K(G)$.

Let G be a strongly connected graph, and let $\mathbf{h}$ be the vector of vertex activities defined in Proposition 3.2. If vertex $v \in V$ is such that $h(v) = 1$ then we say that v is a *simple* vertex of G.

Lemma 7.1. *Let G be a strongly connected graph, and let $v, w \in V(G)$. If w is simple then the Laplacian matrix $Q(G)$ is equivalent to the matrix Q' obtained from Q by replacing every entry in either column v or row w by zero.*

Proof. Let $\mathbf{h}$ be the vector of vertex activities of G. Use elementary column operations to add column u to column v for all $v \neq u \in V$. Then use elementary row operations to add $h(u)$ times row u to row w for all $w \neq u \in V$. The resulting matrix Q' is equivalent to Q and has the required form, since $Q^\dagger \mathbf{h} = Q\mathbf{1} = \mathbf{0}$. ☐

Proposition 7.2. *Let G and H be vertex-disjoint strongly connected graphs, and let $v \in V(G)$ and $w \in V(H)$. Denote by $(G \cup H)/vw$ the graph obtained from $G \cup H$ by identifying v and w. If w is a simple vertex of H then*

$$\mathcal{K}(G \cup H) \simeq \mathcal{K}((G \cup H)/vw) \oplus \mathbf{Z}$$

and

$$K(G \cup H) \simeq K((G \cup H)/vw).$$

Proof. Let M be the submatrix of $Q(G)$ obtained by deleting the row and column indexed by v; so we have

$$Q(G) = \begin{bmatrix} M & \overline{\mathbf{v}} \\ \mathbf{v} & \Delta(G)_{vv} \end{bmatrix}$$

for some row vector $\mathbf{v}$ and column vector $\overline{\mathbf{v}}$. Since $Q(G)\mathbf{1} = \mathbf{0}$, we have

$$Q(G) \approx Q'(G) := \begin{bmatrix} M & \mathbf{0} \\ \mathbf{v} & 0 \end{bmatrix}.$$

Similarly, let N be the submatrix of $Q(H)$ obtained by deleting the row and column indexed by w; so we have

$$Q(H) = \begin{bmatrix} \Delta(H)_{ww} & \mathbf{w} \\ \overline{\mathbf{w}} & N \end{bmatrix}$$

for some row vector $\mathbf{w}$ and column vector $\overline{\mathbf{w}}$. Since w is a simple vertex of H, Lemma 7.1 implies that $Q(H) \approx [0] \oplus N$.

Now, since $Q(G \cup H) = Q(G) \oplus Q(H)$ we see that $Q(G \cup H) \approx Q'(G) \oplus [0] \oplus N$. Also, since w is a simple vertex of H we see that

$$Q((G \cup H)/vw) = \begin{bmatrix} M & \overline{\mathbf{v}} & O \\ \mathbf{v} & d & \mathbf{w} \\ O & \overline{\mathbf{w}} & N \end{bmatrix} \approx \begin{bmatrix} M & \mathbf{0} & O \\ \mathbf{v} & 0 & \mathbf{0} \\ O & \mathbf{0} & N \end{bmatrix} = Q'(G) \oplus N$$

(here, $d := \Delta(G)_{vv} + \Delta(H)_{ww}$). Since $Q(G \cup H) \approx Q((G \cup H)/vw) \oplus [0]$ we have $\mathrm{snf}(G \cup H) = \mathrm{snf}((G \cup H)/vw) \oplus [0]$, from which the result follows. $\square$

Let G and H be vertex-disjoint weakly connected directed graphs, let $v_1 \neq v_2$ be distinct vertices of G, and let $w_1 \neq w_2$ be distinct vertices of H. The graph $G \bullet H := (G \cup H)/\{v_1w_1, v_2w_2\}$ is obtained from $G \cup H$ by identifying v_1 and w_1, and identifying v_2 and w_2. The graph $G \circ H := (G \cup H)/\{v_1w_2, v_2w_1\}$ is obtained from $G \cup H$ by identifying v_1 and w_2, and identifying v_2 and w_1. We say that $G \bullet H$ and $G \circ H$ are related by *twisting a two-vertex cut.*

Proposition 7.3. *Let G and H be vertex-disjoint strongly connected graphs, and use the notation of the above paragraph. If both w_1 and w_2 are simple vertices of H then*

$$\mathcal{K}(G \bullet H) \simeq \mathcal{K}(G \circ H).$$

Proof. For $i = 1, 2$, let $\mathbf{v}_i$ be the row vector with entries $Q(G)_{v_iz}$ for each $z \in V(G) \smallsetminus \{v_1, v_2\}$, and let $\overline{\mathbf{v}}_i$ be the column vector with entries $Q(G)_{zv_i}$ for each $z \in V(G) \smallsetminus \{v_1, v_2\}$. For $i = 1, 2$, let $\mathbf{w}_i$ be the row vector with entries $Q(H)_{w_iz}$ for each $z \in V(H) \smallsetminus \{w_1, w_2\}$, and let $\overline{\mathbf{w}}_i$ be the column vector with entries $Q(H)_{zw_i}$ for each $z \in V(H) \smallsetminus \{w_1, w_2\}$. Then the matrices $Q(G \bullet H)$ and $Q(G \circ H)$ have the block forms as shown:

$$\begin{bmatrix} M & \overline{\mathbf{v}}_1 & \overline{\mathbf{v}}_2 & O \\ \mathbf{v}_1 & a_{11} & a_{12} & \mathbf{w}_1 \\ \mathbf{v}_2 & a_{21} & a_{22} & \mathbf{w}_2 \\ O & \overline{\mathbf{w}}_1 & \overline{\mathbf{w}}_2 & N \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M & \overline{\mathbf{v}}_1 & \overline{\mathbf{v}}_2 & O \\ \mathbf{v}_1 & b_{11} & b_{12} & \mathbf{w}_2 \\ \mathbf{v}_2 & b_{21} & b_{22} & \mathbf{w}_1 \\ O & \overline{\mathbf{w}}_2 & \overline{\mathbf{w}}_1 & N \end{bmatrix}$$

$$Q(G \bullet H) \qquad\qquad Q(G \circ H)$$

Here, M is the submatrix of $Q(G)$ induced by rows and columns in $V(G) \smallsetminus \{v_1, v_2\}$, N is the submatrix of $Q(H)$ induced by rows and columns in $V(H) \smallsetminus \{w_1, w_2\}$, and

$$a_{ij} := Q(G)_{v_i v_j} + Q(H)_{w_i w_j} \quad \text{and} \quad b_{ij} := Q(G)_{v_i v_j} + Q(H)_{w_{3-i}, w_{3-j}}$$

for $1 \leq i \leq 2$.

Since both w_1 and w_2 are simple vertices of H, an argument analogous to the proof of Lemma 7.1 shows that $Q(G \bullet H)$ and $Q(G \circ H)$ are equivalent to

$$\begin{bmatrix} M & \overline{\mathbf{v}}_1 & \mathbf{0} & O \\ \mathbf{v}_1 & a_{11} & 0 & \mathbf{w}_1 \\ \mathbf{v}_1 + \mathbf{v}_2 & c_{21} & 0 & \mathbf{0} \\ O & \overline{\mathbf{w}}_1 & \mathbf{0} & N \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M & \overline{\mathbf{v}}_1 & \mathbf{0} & O \\ \mathbf{v}_1 & b_{11} & 0 & \mathbf{w}_2 \\ \mathbf{v}_1 + \mathbf{v}_2 & c_{21} & 0 & \mathbf{0} \\ O & \overline{\mathbf{w}}_2 & \mathbf{0} & N \end{bmatrix}$$

$$Q'(G \bullet H) \qquad\qquad Q'(G \circ H)$$

respectively, in which $c_{21} = Q(G)_{v_1 v_1} + Q(G)_{v_2 v_1}$.

Let $\mathbf{h}$ denote the vector of activities of H, so that $\mathbf{h}^\dagger Q(H) = \mathbf{0}$. Then the sum over all $z \in V(H) \smallsetminus \{w_1, w_2\}$ of $h(z)$ times the z-th row of N is equal to $-\mathbf{w}_1 - \mathbf{w}_2$, since both w_1 and w_2 are simple in H. Since the columns of $Q(H)$ sum to $\mathbf{0}$, the columns of N sum to $-\overline{\mathbf{w}}_1 - \overline{\mathbf{w}}_2$. Now, for $z \in V(H) \smallsetminus \{w_1, w_2\}$, add $h(z)$ times row z of $Q'(G \circ H)$ to the row indexed by $v_1 w_2$. Then, for $z \in V(H) \smallsetminus \{w_1, w_2\}$, add column z of the resulting matrix to the column indexed by $v_1 w_2$. The result is the matrix

$$Q''(G \circ H) = \begin{bmatrix} M & \overline{\mathbf{v}}_1 & \mathbf{0} & O \\ \mathbf{v}_1 & c_{11} & 0 & -\mathbf{w}_1 \\ \mathbf{v}_1 + \mathbf{v}_2 & c_{21} & 0 & \mathbf{0} \\ O & -\overline{\mathbf{w}}_1 & \mathbf{0} & N \end{bmatrix}$$

in which

$$\begin{aligned}
c_{11} &= b_{11} + \sum_{z \in V(H) \setminus \{w_1,w_2\}} h(z)\overline{w}_2(z) - \sum_{z \in V(H) \setminus \{w_1,w_2\}} w_1(z) \\
&= b_{11} - Q(H)_{w_2w_2} - Q(H)_{w_2w_1} + Q(H)_{w_1w_1} + Q(H)_{w_2w_1} \\
&= b_{11} - \Delta(H)_{w_2w_2} + \Delta(H)_{w_1w_1} \\
&= \Delta(G)_{v_1v_1} + \Delta(H)_{w_2w_2} - \Delta(H)_{w_2w_2} + \Delta(H)_{w_1w_1} \\
&= a_{11}
\end{aligned}$$

Finally, by multiplying the last $n(H)-2$ rows and columns of $Q''(G \circ H)$ by -1, we obtain the matrix $Q'(G \bullet H)$. This shows that $Q(G \bullet H)$ and $Q(G \circ H)$ are equivalent, so that $\mathrm{snf}(Q(G \bullet H)) = \mathrm{snf}(Q(G \circ H))$ and $\mathcal{K}(G \bullet H) \simeq \mathcal{K}(G \circ H)$. □

Example 7.4. The hypothesis in Proposition 7.3 that w_1 and w_2 are simple vertices of H can not be removed, as the following example shows. Let G and H both have Laplacian matrix

$$Q(G) = Q(H) = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix},$$

and note that the vertex activities are 2 and 1 in each graph. Then

$$Q(G \bullet H) = \begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix} \qquad \text{and} \qquad Q(G \circ H) = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix},$$

and by calculating Smith normal forms we see that

$$\mathcal{K}(G \bullet H) \simeq (\mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z} \qquad \text{and} \qquad \mathcal{K}(G \circ H) \simeq (\mathbf{Z}/3\mathbf{Z}) \oplus \mathbf{Z}.$$

A strongly connected directed graph G is *balanced* if, for each vertex $v \in V(G)$, the indegree of v equals the outdegree of v. Undirected graphs are thus a special case of balanced graphs. Equivalently, G is balanced if and only if $Q^\dagger \mathbf{1} = \mathbf{0}$, *i.e.* every vertex of G is simple.

Corollary 7.5. *Let G and H be vertex-disjoint balanced graphs, and let $v \in V(G)$ and $w \in V(H)$. Then*

$$\mathcal{K}(G \cup H) \simeq \mathcal{K}((G \cup H)/vw) \oplus \mathbf{Z}$$

and

$$K(G \cup H) \simeq K((G \cup H)/vw).$$

Corollary 7.6. *Let G and H be vertex-disjoint balanced graphs, let $v_1 \neq v_2$ in $V(G)$, and let $w_1 \neq w_2$ in $V(H)$. With the notation introduced above,*

$$\mathcal{K}(G \bullet H) \simeq \mathcal{K}(G \circ H).$$

Whitney [9] characterized those pairs of undirected graphs G, H which have isomorphic graphic matroids as being exactly those pairs for which G may be transformed into H by some sequence of splittings or mergings of one-vertex cuts and twistings of two-vertex cuts. Corollary 7.7 follows immediately.

Corollary 7.7. *Let G and H be undirected graphs. If the graphic matroids of G and H are isomorphic, then $K(G) \simeq K(H)$.*

Example 7.8. The converse of Corollary 7.7 does not hold. In fact, for undirected graphs, the Tutte polynomial $T(G; x, y)$ is not computable from the critical group $\mathcal{K}(G)$, as the following example shows. Let v be a vertex of C_3 and let w be a vertex of C_4, where the cycles are vertex-disjoint. Then

$$\begin{aligned} \mathcal{K}((C_3 \cup C_4)/vw) &\simeq (\mathbf{Z}/3\mathbf{Z}) \oplus (\mathbf{Z}/4\mathbf{Z}) \oplus \mathbf{Z} \\ &\simeq (\mathbf{Z}/12\mathbf{Z}) \oplus \mathbf{Z} \simeq \mathcal{K}(C_{12}). \end{aligned}$$

However,

$$T((C_3 \cup C_4)/vw; x, y) = (y + x + x^2)(y + x + x^2 + x^3)$$

and

$$T(C_{12}; x, y) = y + x + x^2 + \cdots + x^{11}.$$

Finally, I conjecture that the critical group is not computable from the Tutte polynomial, either.

Conjecture 7.9. *There exist connected undirected graphs G and H such that $T(G; x, y) = T(H; x, y)$ and $\mathcal{K}(G) \not\simeq \mathcal{K}(H)$.*

8. Inequivalence of $\mathcal{T}(G)$ and $K(G)$.

For this section we consider only connected undirected graphs. By Proposition 2.3, in this case the reduced critical group $K(G)$ is a finite abelian group of order $\kappa(G)$, the cardinality of the set $\mathcal{T}(G)$ of spanning trees of G. It is thus reasonable to consider the problem of constructing a bijection from $\mathcal{T}(G)$ to $K(G)$ in a 'natural' way. We will show that, in general, this is not possible. To allow more flexibility in the construction, we consider the complex vector spaces $\mathbf{C}\mathcal{T}(G)$ and $\mathbf{C}K(G)$, and ask for an isomorphism of vector spaces $\psi_G : \mathbf{C}\mathcal{T}(G) \to \mathbf{C}K(G)$ which is constructed naturally from G. The naturality condition means that ψ_G should depend only on the isomorphism class of G, but we must first state this more precisely.

For any graph isomorphism $f : G \to H$, there is an induced bijection $f_{\mathcal{T}} : \mathcal{T}(G) \to \mathcal{T}(H)$ defined by sending each spanning tree of G to its image under f. This extends linearly to an isomorphism from $\mathbf{C}\mathcal{T}(G)$

to $\mathbf{C}\mathfrak{T}(H)$ which we also denote by $f_{\mathfrak{T}}$. Let Φ be the matrix indexed by $V(H) \times V(G)$, with

$$\Phi_{vw} := \begin{cases} 1 & \text{if } f(w) = v, \\ 0 & \text{if } f(w) \neq v. \end{cases}$$

Then $\Phi : \mathbf{Z}^{V(G)} \to \mathbf{Z}^{V(H)}$ is an isomorphism, and $\Phi Q^\dagger(G) = Q^\dagger(H)\Phi$. It follows that $\Phi Q^\dagger(G)\mathbf{Z}^{V(G)} = Q^\dagger(H)\mathbf{Z}^{V(H)}$, and so f induces a group isomorphism

$$f_{\mathcal{K}} : \mathcal{K}(G) \xrightarrow{\sim} \mathcal{K}(H)$$

well-defined by $f_{\mathcal{K}}(\mathbf{x} + Q^\dagger(G)\mathbf{Z}^{V(G)}) := \Phi\mathbf{x} + Q^\dagger(H)\mathbf{Z}^{V(H)}$. For reduced critical groups, we also have a group isomorphism

$$f_K : K(G) \xrightarrow{\sim} K(H)$$

induced by f, and we extend this linearly to obtain an isomorphism from $\mathbf{C}K(G)$ to $\mathbf{C}K(H)$, also denoted by f_K.

The naturality condition on ψ_G is that, for any multigraph isomorphism $f : G \to H$, the diagram

$$\begin{array}{ccc} \mathbf{C}\mathfrak{T}(G) & \xrightarrow{\psi_G} & \mathbf{C}K(G) \\ {\scriptstyle f_{\mathfrak{T}}} \downarrow & & \downarrow {\scriptstyle f_K} \\ \mathbf{C}\mathfrak{T}(H) & \xrightarrow{\psi_H} & \mathbf{C}K(H) \end{array}$$

is commutative; that is to say, $\psi_H \circ f_{\mathfrak{T}} = f_K \circ \psi_G$. This means that ψ_H is 'the same as' ψ_G, after relabelling the vertices according to f.

Theorem 8.1. *There exist connected undirected graphs G for which there is no natural isomorphism $\psi_G : \mathbf{C}\mathfrak{T}(G) \to \mathbf{C}K(G)$.*

Proof. In particular, the naturality condition must hold when $G = H$ and f is in the automorphism group $\mathrm{Aut}(G)$ of G. In this situation, the assignment $f \mapsto f_{\mathfrak{T}}$ gives a representation of $\mathrm{Aut}(G)$ acting on $\mathbf{C}\mathfrak{T}(G)$ and the assignment $f \mapsto f_K$ gives a representation of $\mathrm{Aut}(G)$ acting on $\mathbf{C}K(G)$. (All the representation theory we need is in Chapter 1 of Ledermann [7].) Commutativity of the diagram means that ψ_G is an $\mathrm{Aut}(G)$-equivariant isomorphism, so these two representations are linearly equivalent. A representation of a finite group is determined up to linear equivalence by its group character, so ψ_G exists if and only if the characters $\chi_{\mathfrak{T}}$ and χ_K of these two representations are equal. Since these are permutation representations, their characters are given by counting fixed points; that is, for each $f \in \mathrm{Aut}(G)$,

$$\chi_{\mathfrak{T}}(f) = \#\{T \in \mathfrak{T}(G) : \ f_{\mathfrak{T}}(T) = T\}$$

and

$$\chi_K(f) = \#\{\mathbf{x} \in K(G) : \ f_K(\mathbf{x}) = \mathbf{x}\},$$

respectively.

Thus, to show that a natural construction of ψ_G is impossible, it suffices to find a connected undirected graph G and automorphism $f \in \mathrm{Aut}(G)$ such that $\chi_{\mathcal{T}}(f) \neq \chi_K(f)$. This is easy: let G be a circulant graph with at least three vertices, and let $f \in \mathrm{Aut}(G)$ be a cyclic permutation of all of $V(G)$. Every spanning tree of G has both leaves and non-leaf vertices, and so it can not be left fixed by f. Thus, $\chi_{\mathcal{T}}(f) = 0$. On the other hand, the $\mathbf{0}$ element of $K(G)$ is such that $f_K(\mathbf{0}) = \mathbf{0}$, since f_K is a group automorphism. Thus, $\chi_K(f) \geq 1$, and since $\chi_{\mathcal{T}} \neq \chi_K$ it follows that ψ_G does not exist. □

I have not thought much about the following problem, but the question seems interesting.

Problem 8.2. With notation as in the proof of Theorem 8.1, does there exist a vertex-transitive connected undirected graph G, with at least three vertices, such that the characters $\chi_{\mathcal{T}}$ and χ_K of $\mathrm{Aut}(G)$ are equal?

Note, in particular, that for such a graph every automorphism fixes at least one spanning tree; together with vertex-transitivity, this seems to be quite restrictive.

In contrast with Theorem 8.1, a negative result, if one fixes a total order $\prec$ on the edge-set of G then the automorphism group of the pair $(G, \prec)$ is trivial (as long as G has at least three vertices), so the naturality condition for these structures becomes vacuous and the possibility arises of constructing a bijection from $\mathcal{T}(G)$ to $K(G)$ relative to $\prec$. Jonathan Dumas (personal communication, August 2000) has recently found such a construction which, moreover, behaves well regarding the external activities of spanning trees with respect to $\prec$.

9. The dollar game for strongly connected graphs.

For undirected graphs, a combinatorial understanding of the critical group has been developed by Biggs [2, 3, 4]. We generalize this to the case of strongly connected graphs; the theory is essentially the same as in the undirected case, with one interesting extra complication.

Let $G = (V, E)$ be a strongly connected graph with no loops, and let \$ denote a designated vertex of G, which we call the *bank*. A *configuration* is an integer vector $\mathbf{c} \in \mathbf{Z}^V$ such that $\mathbf{1}^\dagger \mathbf{c} = 0$, so that $c(\$) = -\sum_{v \neq \$} c(v)$. We say that a configuration $\mathbf{c}$ is *nonnegative* when $c(v) \geq 0$ for all $\$ \neq v \in V$. A vertex $v \neq \$$ is *legal* for $\mathbf{c}$ when $c(v) \geq \Delta_{vv}$, the outdegree of v. The configuration $\mathbf{c}$ is *stable* when $c(v) < \Delta_{vv}$ for all $v \neq \$$; in this case, and in this case only, the bank

vertex \$ is *legal* for $\mathbf{c}$. For a configuration $\mathbf{c}$ and vertex v, the effect of *firing v from* $\mathbf{c}$ is the configuration $\mathbf{c}|v := \mathbf{c} - Q^\dagger \hat{v}$, in which $\hat{v} \in \mathbf{Z}^V$ denotes the characteristic vector of the vertex $v \in V$. More explicitly, for each $w \in V$, $(c|v)(w)$ is defined by

$$(c|v)(w) = \begin{cases} c(v) - \Delta_{vv} & \text{if } w = v, \\ c(w) + A_{vw} & \text{if } w \neq v. \end{cases}$$

If one considers a nonnegative configuration $\mathbf{c}$ as representing $c(v)$ dollars at each vertex $v \neq \$$, then the effect of firing a legal vertex v is to send one dollar along each edge with initial vertex v. More generally, if $v_1 \cdots v_k$ is a sequence of vertices in which $v \in V$ occurs $m(v)$ times, then the effect of firing this sequence of vertices from the configuration $\mathbf{c}$ is $\mathbf{c}|v_1 \cdots v_k = \mathbf{c} - Q^\dagger \mathbf{m}$, in which $\mathbf{m} = \hat{v}_1 + \cdots + \hat{v}_k$ is the vector of multiplicities. The sequence $v_1 \cdots v_k$ is *legal for* $\mathbf{c}$ provided that v_i is legal for $\mathbf{c}|v_1 \cdots v_{i-1}$ for each $1 \leq i \leq k$. For a configuration $\mathbf{c}$, let $\mathcal{S}(\mathbf{c})$ denote the set of all configurations $\mathbf{b}$ such that $\mathbf{b} = \mathbf{c}|v_1 \cdots v_k$ for some sequence $v_1 \cdots v_k$ of vertices which is legal for $\mathbf{c}$ and does not contain the bank vertex \$.

Proposition 9.1. *Let $G = (V, E)$ be a loopless strongly connected graph with bank vertex \$. For every configuration $\mathbf{c}$ on $(G, \$)$, the set $\mathcal{S}(\mathbf{c})$ is finite. If $\mathbf{c}$ is nonnegative then every configuration in $\mathcal{S}(\mathbf{c})$ is also nonnegative.*

Proof. If $\mathbf{c}$ is nonnegative and v is legal for $\mathbf{c}$, then $\mathbf{c}|v$ is nonnegative. From this it follows that if $\mathbf{c}$ is nonnegative then every configuration in $\mathcal{S}(\mathbf{c})$ is nonnegative. More generally, for any configuration $\mathbf{c}$, define the configuration $\mathbf{c}^-$ by

$$c^-(v) := \begin{cases} c(v) & \text{if } v \neq \$ \text{ and } c(v) < 0, \\ 0 & \text{if } v \neq \$ \text{ and } c(v) \geq 0, \end{cases}$$

and $c^-(\$) := -\sum_{v \neq \$} c^-(v)$. If $v_1 \cdots v_k$ is a legal sequence for $\mathbf{c}$ which does not contain \$, then it is a legal sequence for $\mathbf{c} - \mathbf{c}^-$. Since $\mathbf{c} - \mathbf{c}^-$ is nonnegative, it follows that $\mathbf{c}|v_1 \cdots v_k - \mathbf{c}^-$ is nonnegative. That is, $\mathbf{b} - \mathbf{c}^-$ is nonnegative for all $\mathbf{b} \in S(\mathbf{c})$.

For each $v \in V$, let $d(v)$ denote the length of a shortest directed path from v to \$ in G, and let $r := \max\{d(v) : \ v \in V\}$. For each configuration $\mathbf{c}$, define the 'label' of $\mathbf{c}$ to be $\ell(\mathbf{c}) := (\ell_1, \ldots, \ell_r)$, in which $\ell_i := \sum\{c(v) : \ v \in V \text{ and } d(v) = i\}$. Define a total order $\prec$ on $\mathbf{Z}^r$ as follows: $(p_1, \ldots, p_r) \prec (q_1, \ldots, q_r)$ if and only if either $p_1 + \cdots + p_r < q_1 + \cdots + q_r$, or $p_1 + \cdots + p_r = q_1 + \cdots + q_r$ and $p_1 = q_1$, $p_2 = q_2, \ldots\ p_{i-1} = q_{i-1}$, $p_i > q_i$ for some $1 \leq i \leq r$. Notice that $(\mathbf{Z}^r, \prec)$ has the same order type as $(\mathbf{Z}, <)$. Also notice that if $v \neq \$$ and v is

legal for $\mathbf{c}$, then

$$\ell(\mathbf{c}^-) \preceq \ell((\mathbf{c}|v)^-) \preceq \ell(\mathbf{c}|v) \prec \ell(\mathbf{c}).$$

It follows that, for any configuration $\mathbf{c}$, the set $\{\ell(\mathbf{b}) : \mathbf{b} \in \mathcal{S}(\mathbf{c})\}$ is finite. But for any $\mathbf{q} \in \mathbf{Z}^r$, the set of configurations $\{\mathbf{b} : \ell(\mathbf{b}) = \mathbf{q}\}$ is also finite. These two observations suffice to show that $\mathcal{S}(\mathbf{c})$ is finite. $\square$

Lemma 9.2. *Let $G = (V, E)$ be a loopless strongly connected graph with bank vertex \$. Let $\mathbf{c}$ be a configuration on $(G, \$)$, let $v, w \in V$, and let $\mathbf{c}|w := \mathbf{c} - Q^\dagger \hat{w}$. If $v \neq w$ then $(c|w)(v) \geq c(v)$. In particular, if $v \notin \{w, \$\}$ and v is legal for $\mathbf{c}$, then v is legal for $\mathbf{c}|w$.*

Proof. This follows immediately from the facts that the off-diagonal elements of Q are nonpositive, and that a vertex v is legal for a configuration $\mathbf{c}$ if and only if $c(v) \geq \Delta_{vv}$. $\square$

Lemma 9.3. *Let $G = (V, E)$ be a loopless strongly connected graph with bank vertex \$. Let $\mathbf{c}$ be a configuration on $(G, \$)$, let $\mathbf{m} \in \mathbf{N}^V$, and let $v_1 \cdots v_k$ be a sequence of vertices, with multiplicity vector $\mathbf{n} := \hat{v}_1 + \cdots + \hat{v}_k$. Produce the sequence $w_1 \cdots w_\ell$ by deleting the first $\min\{m(z), n(z)\}$ occurrences of vertex z from the sequence $v_1 \cdots v_k$, for each $z \in V$. If $v_1 \cdots v_k$ is legal for $\mathbf{c}$ and $n(\$) = 0$, then $w_1 \cdots w_\ell$ is legal for $\mathbf{c}' := \mathbf{c} - Q^\dagger \mathbf{m}$.*

Proof. We proceed by induction on k, the length of $v_1 \cdots v_k$.

For the basis of induction, $k = 1$, assume that $v \neq \$$ is legal for $\mathbf{c}$. If $m(v) \geq 1$ then the empty sequence is legal for $\mathbf{c}'$, as required. Otherwise, write $\mathbf{m} = \hat{u}_1 + \cdots + \hat{u}_r$ for some sequence of vertices $u_1 \cdots u_r$. Since $m(v) = 0$, v does not occur in the sequence $u_1 \cdots u_r$. The previous lemma and induction on r now show that v is legal for $\mathbf{c}'$, as required.

For the induction step, assume the result for sequences of length $k-1$, and consider $v_1 \cdots v_k$. First, assume that $m(v_1) \geq 1$, and consider $\mathbf{b} := \mathbf{c}|v_1$ and $\mathbf{m}' := m - \hat{v}_1$. Then $\mathbf{c}' = \mathbf{b} - Q^\dagger \mathbf{m}'$ and $v_2 \cdots v_k$ is legal for $\mathbf{b}$. Applying the induction hypothesis to $\mathbf{b}$, $\mathbf{m}'$, and $v_2 \cdots v_k$, we see that $w_1 \cdots w_\ell$ is legal for $\mathbf{c}'$. For the remaining case, assume that $m(v_1) = 0$, so that $w_1 = v_1$. As in the basis of induction, since v_1 is legal for $\mathbf{c}$, v_1 is legal for $\mathbf{c}'$. Now apply the induction hypothesis to $\mathbf{b} := \mathbf{c}|v_1$, $\mathbf{m}$, and $v_2 \cdots v_k$ to conclude that $w_2 \cdots w_\ell$ is legal for $\mathbf{c}'|v_1$. Hence, $w_1 \cdots w_\ell$ is legal for $\mathbf{c}'$, completing the induction step and the proof. $\square$

Proposition 9.4. *Let $G = (V, E)$ be a loopless strongly connected graph with bank vertex \$. For every configuration $\mathbf{c}$ on $(G, \$)$, the set $\mathcal{S}(\mathbf{c})$ contains a unique stable configuration.*

Proof. Let $\mathcal{D}$ be the graph with vertex-set $\mathcal{S}(\mathbf{c})$ and directed edges $\mathbf{b} \rightarrow \mathbf{b}|v$ when $v \neq \$$ is legal for $\mathbf{b} \in \mathcal{S}(\mathbf{c})$. Then $\mathcal{D}$ is a nonempty graph, and, by the proof of Proposition 9.1, since $\ell(\mathbf{b}|v) \prec \ell(\mathbf{b})$ for all $\mathbf{b} \in \mathcal{S}(\mathbf{c})$ $\mathcal{D}$ contains no directed cycles. Therefore, $\mathcal{D}$ has at least one sink vertex, which is a stable configuration on $(G, \$)$.

Now suppose that $\mathbf{a}$ and $\mathbf{b}$ are two stable configurations in $\mathcal{S}(\mathbf{c})$. Let $v_1 \cdots v_k$ and $u_1 \cdots u_r$ be sequences of vertices which are legal for $\mathbf{c}$, do not contain \$, and are such that $\mathbf{c}|v_1 \cdots v_k = \mathbf{a}$ and $\mathbf{c}|u_1 \cdots u_r = \mathbf{b}$. Let $\mathbf{n} := \hat{v}_1 + \cdots + \hat{v}_k$ and let $\mathbf{m} := \hat{u}_1 + \cdots + \hat{u}_r$. The hypothesis of Lemma 9.3 is satisfied, so produce the subsequence $w_1 \cdots w_\ell$ of $v_1 \cdots v_k$ as in that lemma. Now, since $w_1 \cdots w_\ell$ is a legal sequence for $\mathbf{c} - Q^\dagger \mathbf{m} = \mathbf{c}|u_1 \cdots u_r = \mathbf{b}$ which does not contain \$, and since $\mathbf{b}$ is stable, it follows that $w_1 \cdots w_\ell$ is the empty sequence. From the construction of $w_1 \cdots w_\ell$, it follows that $m(z) \geq n(z)$ for each $z \in V$. By symmetry, we may repeat this argument with $v_1 \cdots v_k$ and $u_1 \cdots u_r$ interchanged, and deduce that $n(z) \geq m(z)$ for each $z \in V$. Finally, since $\mathbf{m} = \mathbf{n}$ we conclude that $\mathbf{a} = \mathbf{c} - Q^\dagger \mathbf{n} = \mathbf{c} - Q^\dagger \mathbf{m} = \mathbf{b}$, finishing the proof. □

We define the *stabilization* of a configuration $\mathbf{c}$ to be the unique stable configuration in $\mathcal{S}(\mathbf{c})$. If $\mathbf{c}$ is a stable configuration, then we define the *successor* $\sigma(\mathbf{c})$ of $\mathbf{c}$ to be the stabilization of $\mathbf{c}|\$$. Thus, σ is an endofunction on the set of stable configurations on $(G, \$)$. We say that a stable configuration $\mathbf{c}$ is *critical* when $\sigma^k(\mathbf{c}) = \mathbf{c}$ for some positive integer k. (Here, σ^k denotes the m-th functional iterate of σ.)

Lemma 9.5. *Let $G = (V, E)$ be a loopless strongly connected graph with bank vertex \$.*
(a) *Every critical configuration on $(G, \$)$ is nonnegative.*
(b) *For every stable configuration $\mathbf{c}$ on $(G, \$)$, there is a nonnegative integer m such that $\sigma^m(\mathbf{c})$ is nonnegative.*
(b) *For every stable configuration $\mathbf{c}$ on $(G, \$)$, there is a nonnegative integer m such that $\sigma^m(\mathbf{c})$ is critical.*

Proof. For part (a), let $\mathbf{c}$ be a critical configuration, and let $v_1 \cdots v_k$ be a nonempty legal sequence of vertices for $\mathbf{c}$, such that $\mathbf{c}|v_1 \cdots v_k = \mathbf{c}$. Then $\mathbf{n} := \hat{v}_1 + \cdots + \hat{v}_k$ is a nonzero vector of nonnegative integers such that $\mathbf{c} = \mathbf{c} - Q^\dagger \mathbf{n}$, so that $Q^\dagger \mathbf{n} = \mathbf{0}$. By Proposition 3.2, it follows that $\mathbf{n} = \lambda \mathbf{h}$ is a positive integer multiple of the vector $\mathbf{h}$ of vertex activities. Every $v \in V$ is fired in the sequence $v_1 \cdots v_k$ exactly $\lambda h(v)$ times, hence at least once, and so $c(v) \geq 0$ for all $v \neq \$$.

For part (b), we use the strategy of the proof of Proposition 9.1, but with a different definition of the label of a configuration. For a configuration $\mathbf{c}$, define $\mathbf{c}^-$ as in the proof of Proposition 9.1. If $\mathbf{c}^- \neq \mathbf{0}$, then for each $v \neq \$$, let $d(v)$ denote the length of a shortest directed path which begins at v and ends at a vertex $w \neq \$$ such that $c(w) < 0$, and let r be the maximum value of $d(v)$ for all $v \neq \$$ such that $c(v) \geq 0$. For $1 \leq j \leq r$, let $\ell_j := \sum\{c(v) : \ d(v) = j\}$, and define the 'label' of $\mathbf{c}$ to be $\ell(\mathbf{c}) := (\mathbf{c}^-, \ell_1, \ldots, \ell_r)$. Define a partial order on the set of all labels as follows: $(\mathbf{a}, k_1, \ldots, k_s) \prec (\mathbf{b}, \ell_1, \ldots, \ell_r)$ if either $\mathbf{b} \neq \mathbf{a}$ and $\mathbf{b} - \mathbf{a}$ is nonnegative (except at \$), or $\mathbf{b} = \mathbf{a} \neq \mathbf{0}$ (in which case $s = r > 0$) and $k_1 = \ell_1$, $k_2 = \ell_2$, $\ldots$, $k_{j-1} = \ell_{j-1}$, $k_j < \ell_j$ for some $1 \leq j \leq r$.

Notice that for any configuration $\mathbf{a}$ with $-\mathbf{a}$ nonnegative, there are only finitely many stable configurations $\mathbf{c}$ such that $\mathbf{c}^- = \mathbf{a}$ (in fact, the number of them is $\prod\{\Delta_{vv} : \ v \neq \ \$ \text{ and } a(v) = 0\}$). It follows that the set of all stable configurations, partially ordered by the order $\prec$ on their labels, has no infinite ascending chains. Now, if v is legal for $\mathbf{c}$ and $\mathbf{c}^- \neq \mathbf{0}$, then $\ell(\mathbf{c}) \prec \ell(\mathbf{c}|v)$, even when $v = \$$. Hence, if $\mathbf{c}$ is stable but not nonnegative, then $\ell(\mathbf{c}) \prec \ell(\sigma(\mathbf{c}))$. Since there are no infinite ascending chains, there is a nonnegative integer m such that $\sigma^m(\mathbf{c})$ is maximal, and this must be a stable, nonnegative configuration.

For part (c), let $\mathbf{c}$ be any stable configuration. By part (b), we may assume that $\mathbf{c}$ is nonnegative. There are exactly $\prod\{\Delta_{vv} : v \neq \$\}$ nonnegative stable configurations on $(G, \$)$, and the successor function acts as an endofunction on this set. Since this set is finite, for any $\mathbf{c}$ in it there exists a nonnegative integer m and positive integer k such that $\sigma^{m+k}(\mathbf{c}) = \sigma^m(\mathbf{c})$. That is, $\sigma^m(\mathbf{c})$ is critical. $\square$

Let $\mathcal{C}(G, \$)$ denote the set of all critical configurations of $(G, \$)$. For critical configurations $\mathbf{a}$ and $\mathbf{b}$ on $(G, \$)$, we say that $\mathbf{a}$ and $\mathbf{b}$ are *coeval* if $\sigma^m(\mathbf{a}) = \mathbf{b}$ for some nonnegative integer m.

Lemma 9.6. *Let $G = (V, E)$ be a loopless strongly connected graph with bank vertex \$. Coevalence is an equivalence relation on $\mathcal{C}(G, \$)$, and each Coevalence class has cardinality divisible by $h(\$)$, in which $\mathbf{h}$ is the vector of vertex activities.*

Proof. That coevalence is an equivalence relation is easy to see. Let $\mathbf{c}$ be a critical configuration on $(G, \$)$, let $v_1 \cdots v_k$ be a nonempty sequence of vertices which is legal for $\mathbf{c}$, such that $\mathbf{c}|v_1 \cdots v_k = \mathbf{c}$, and as short as possible subject to these conditions, and let $\mathbf{n} := \hat{v}_1 + \cdots + \hat{v}_k$. As in the proof of Lemma 9.5(a), we see that $\mathbf{n} = \lambda \mathbf{h}$ for some positive integer λ. Since the bank vertex occurs $\lambda h(\$)$ times in the sequence $v_1 \cdots v_k$ it follows that $\mathbf{c}$ is coeval with exactly $\lambda h(\$)$ critical configurations. $\square$

Proposition 9.7. *Let $G = (V, E)$ be a loopless strongly connected graph, and let \$ be a simple vertex of G. Then every coevalence class of $\mathcal{C}(G, \$)$ is a singleton.*

Proof. Let $\mathbf{h}$ be the vector of vertex activities of G. Let $\mathbf{c}$ be any critical configuration of $(G, \$)$, and let $v_1 \cdots v_k$ be a nonempty sequence of vertices which is legal for $\mathbf{c}$ and such that $\mathbf{c}|v_1 \cdots v_k = \mathbf{c}$. Let $\mathbf{n} := \hat{v}_1 + \cdots + \hat{v}_k$ be the multiplicity vector of $v_1 \cdots v_k$; as in Lemma 9.5(a) we have $\mathbf{n} = \lambda \mathbf{h}$ for some positive integer λ.

Since $\mathbf{c}$ is stable, $v_1 = \$$. If this is the only occurrence of \$ in the sequence $v_1 \cdots v_k$, then $\sigma(\mathbf{c}) = \mathbf{c}$ and it follows that the coevalence class of $\mathbf{c}$ is the singleton $\{\mathbf{c}\}$. Otherwise, let $1 = p_1 < p_2 < \cdots < p_r \leq k$ be all the indices from 1 to k such that $v_{p_i} = \$$, and let $p_{r+1} = k + 1$. Since $\mathbf{n} = \lambda \mathbf{h}$ and $h(\$) = 1$, we see that $\lambda = r$.

Now, $v_2 \cdots v_{p_2 - 1}$ is a legal sequence for $\mathbf{c}|\$$ which does not contain any occurrence of \$. Applying Lemma 9.3 with $\mathbf{m} = \mathbf{h} - \hat{\$}$, let $w_1 \cdots w_\ell$ be the sequence so produced. This sequence is legal for $(\mathbf{c}|\$) - Q^\dagger \mathbf{m} = \mathbf{c} - Q^\dagger \mathbf{h} = \mathbf{c}$, and since $\mathbf{c}$ is stable, it follows that $w_1 \cdots w_\ell$ is the empty sequence. Therefore, for each $\$ \neq u \in V$, the multiplicity of u in $v_2 \cdots v_{p_2-1}$ is at most $h(u)$.

Since $\mathbf{b} := \mathbf{c}|\$ v_2 \cdots v_{p_2-1}$ is critical, we may repeat the argument of the above paragraph, using $\mathbf{b}|\$$ and $v_{p_2+1} \cdots v_{p_3-1}$. As above, we conclude that for each $\$ \neq u \in V$, the multiplicity of u in $v_{p_2+1} \cdots v_{p_3-1}$ is at most $h(u)$. For each $1 \leq i \leq r$, let $\mathbf{n}_i := \hat{v}_{p_i+1} + \cdots + \hat{v}_{p_{i+1}-1}$. Applying the above argument for each sequence $v_{p_i+1} \cdots v_{p_{i+1}-1}$, we see that $n_i(u) \leq h(u)$ for each $1 \leq i \leq r$ and $\$ \neq u \in V$. But $\mathbf{n}_1 + \cdots + \mathbf{n}_r + r\hat{\$} = \mathbf{n} = r\mathbf{h}$, from which it follows that $\mathbf{n}_1 = \cdots = \mathbf{n}_r = \mathbf{h} - \hat{\$}$. Therefore, $\mathbf{c}|\$ v_2 \cdots v_{p_2-1} = \mathbf{c} - Q^\dagger \mathbf{h} = \mathbf{c}$, so that $\sigma(\mathbf{c}) = \mathbf{c}$, and it follows that the coevalence class of $\mathbf{c}$ is the singleton $\{\mathbf{c}\}$, completing the proof. $\square$

Example 9.8. As the following example shows, if \$ is not a simple vertex of G, then it may happen that some coevalence class of $\mathcal{C}(G, \$)$ has cardinality strictly greater than $h(\$)$. Consider the graph G with Laplacian matrix and vector of vertex activities

$$Q = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{h} = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 4 \end{bmatrix}.$$

Denoting the vertices by 1, 2, 3, 4 corresponding to the matrix indices, take $\$ = 1$ for the bank vertex. Denoting a configuration $\mathbf{c}$ for this

$(G,\$)$ by the triple $c(2)c(3)c(4)$, we see that the nonnegative stable configurations are mapped by the successor function as

$$\begin{array}{ccccccccccccc}
000 & & & & 101 & & & & & & & & \\
\downarrow & & & & \downarrow & & & & & & & & \\
110 & \to & 100 & \to & 011 & \to & 001 & \to & 111 & \to & 101 & \to & 110.
\end{array}$$

Thus, $\mathcal{C}(G,\$)$ has a single coevalence class of cardinality six.

Finally, we connect the critical group of G with the results of this section; this should be compared with Theorems 3.8 and 8.1 of Biggs [4]. Let $\mathbf{c}$ be any configuration on $(G,\$)$, and let $\overline{\mathbf{c}}$ be the stabilization of $\mathbf{c}$. From Lemma 9.5(c), there is a nonnegative integer m_0 such that $\sigma^m(\overline{\mathbf{c}})$ is critical for all $m \geq m_0$. These critical configurations are all coeval with one another, and this coevalence class is determined uniquely by $\mathbf{c}$; we denote it by $[\mathbf{c}]$.

Theorem 9.9. *Let $G = (V,E)$ be a loopless strongly connected graph with bank vertex $\$$, and let $U \subseteq \mathbf{Z}^V$ consist of those vectors $\mathbf{u} \in \mathbf{Z}^V$ such that $\mathbf{1}^\dagger \mathbf{u} = 0$. Then $K(G)$ is the subgroup $U/Q^\dagger\mathbf{Z}^V$ of $\mathcal{K}(G)$, and the elements of $K(G)$ correspond bijectively with coevalence classes of critical configurations on $(G,\$)$* via *the correspondence* $\mathbf{u}+Q^\dagger\mathbf{Z}^V \mapsto [\mathbf{u}]$.

Proof. For $\mathbf{x} \in \mathbf{Z}^V$ the element $\mathbf{z} + Q^\dagger\mathbf{Z}^V$ of $\mathcal{K}(G)$ is in $K(G)$ if and only if $\lambda\mathbf{z} \in Q^\dagger\mathbf{Z}^V$ for some positive integer λ. Since every column $\mathbf{q}$ of $Q^\dagger$ satisfies $\mathbf{1}^\dagger\mathbf{q} = 0$, if $\mathbf{z} + Q^\dagger\mathbf{Z}^V$ is in $K(G)$ then $\mathbf{1}^\dagger\mathbf{z} = 0$. Conversely, if $\mathbf{1}^\dagger\mathbf{z} = 0$ then $\mathbf{z} = Q^\dagger\mathbf{w}$ for some rational V-indexed vector $\mathbf{w}$, since the rank of $Q^\dagger$ is $n(G)-1$. Hence, $\lambda\mathbf{z} \in Q^\dagger\mathbf{Z}^V$ for some positive integer λ, so that $\mathbf{z} + Q^\dagger\mathbf{Z}^V$ is in $K(G)$. This proves that $K(G) = U/Q^\dagger\mathbf{Z}^V$.

For the second claim, consider two configurations $\mathbf{b}$ and $\mathbf{c}$ on $(G,\$)$, so that $\mathbf{b},\mathbf{c} \in U$. If $[\mathbf{b}] = [\mathbf{c}]$ then there are nonnegative integers p, q such that $\sigma^p(\overline{\mathbf{b}}) = \sigma^q(\overline{\mathbf{c}})$. Therefore, there are V-indexed vectors $\mathbf{m},\mathbf{n}$ of nonnegative integers such that $\mathbf{b} - Q^\dagger\mathbf{m} = \mathbf{c} - Q^\dagger\mathbf{n}$, so that $\mathbf{b} - \mathbf{c}$ is in $Q^\dagger\mathbf{Z}^V$. This proves that the map $\mathbf{u} + Q^\dagger\mathbf{Z}^V \mapsto [\mathbf{u}]$ is injective. Since this map is clearly surjective, the theorem is proved. □

Problem 9.10. Given a strongly connected graph G and bank vertex $\$ \in V$, say that $\$$ is *small* when every coevalence class of $\mathcal{C}(G,\$)$ has cardinality $h(\$)$, and that $\$$ is *fair* when all coevalence classes of $\mathcal{C}(G,v)$ have equal cardinality. Are there polynomial-time algorithms for determining whether a given vertex is small, or is fair?

References

[1] R. Bacher, P. de la Harpe, and T. Nagnibeda, The lattice of integral flows and the lattice of integral coboundaries on a finite graph, *Bull. Soc. Math. de France* **125** (1997), 167-198.

[2] N.L. Biggs, "Algebraic Graph Theory," 2nd Edition, Cambridge University Press, Cambridge, 1993.

[3] N.L. Biggs, Algebraic potential theory on graphs, *Bull. London Math. Soc.* **29** (1997), 641-682.

[4] N.L. Biggs, Chip-firing and the critical group of a graph, *J. Algebraic Combin.* **9** (1999), 25-46.

[5] N.L. Biggs, The Tutte polynomial as a growth function, *J. Algebraic Combin.* **10** (1999), 115-134.

[6] C.D. Godsil, "Algebraic Combinatorics," Chapman & Hall, New York / London, 1993.

[7] W. Ledermann, "Introduction to Group Characters," Cambridge University Press, Cambridge, 1977.

[8] E.M. Palmer, "Graphical Evolution: An Introduction to the Theory of Random Graphs," John Wiley & Sons, New York, 1985.

[9] H. Whitney, 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 245-254.

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